

Wave function as geometric entity

B. I. Lev

Bogolyubov Institute for Theoretical Physics of the NAS of Ukraine, 14-b, Metrolohichna Str., Kyiv 03680, Ukraine

(Dated: January 12, 2013)

A new approach to the geometrization of the electron theory is proposed. The particle wave function is represented by a geometric entity, i.e., Clifford number, with the translation rules possessing the structure of Dirac equation for any manifold. A solution of this equation is obtained in terms of geometric treatment. Interference of electrons whose wave functions are represented by geometric entities is considered. New experiments concerning the geometric nature of electrons are proposed.

PACS numbers: 73.21.Fg, 78.67.De

The problem of how to geometrize the electron theory and include it in the scheme of the general relativity theory is far from being solved. The expression for the covariant derivative obtained in [1] from intuition consideration, with some interpretational corrections introduced in [2], is the generally accepted formula now. Cartan [3] has shown, however, that finite-dimensional representation of a complete linear group of coordinate transformations does not exist. Moreover, the set of Dirac spinors preserves the structure of the linear vector space, but does not preserve the ring structure since defining the composition operation involves some complications. Thus allowed states are depleted inasmuch as wave function behavior under the parallel translation cannot be calculated and appropriately interpreted, and, besides that, the states of the particle ensemble cannot be determined.

A way to solve the problem is proposed in our paper. We employ the idea of correspondence between the spinor matrices and the elements of an exterior algebra and thus define the space of states in terms of a space of representations of a space-time Clifford algebra. Then the particle wave function is represented by a complete geometric entity - a sum of probable direct forms of an induced space of the Clifford algebra. This algebra possesses ring structure [5] since it is a vector space over the field of real numbers and hence makes an additive group whose low of elements composition is distributive rather than commutative with respect to addition. This ring has ideals which may be obtained by multiplying the separated element on the right or on the left by ring elements [5]. The ideals resulting from this procedure are just the Dirac spinors of the standard approach. Thus the representation of the Clifford algebra by the Clifford number contains more information on particle properties than spinor representations. Moreover, having attributed the wave function with geometrical sense we can obtain correct translation rules for an arbitrary manifold [7] and come to some new quantum results associated with the geometric nature of the wave function. Among these results we indicate the observation that Dirac equation in the geometric representation is nothing but the translation equation in the general relativity sense, hence its solutions may be interpreted geometrically. Moreover, the geometric representation of the wave function yields

other results concerning the interference of elementary particles which just may reveal the geometric nature of the wave function [8]. We can also predict a new effect, i.e., time delay of elementary-particle tunneling through a potential barrier which can be explained only in terms of the approach proposed [9].

We can introduce, at each point of the manifold, an arbitrary basis that corresponds to the vector basis, and construct an induced space of all possible products of basis vectors. Making use of Clifford algebra for the composition of individual vectors with simultaneously existing both inner and outer products, we can set, at each point of the manifold, a unique complete geometric characteristic - the direct sum of all possible products for the elements of the induced space. The direct sum of such tensor representations can be attributed with the Clifford algebra structure by means of the direct product [5]. The final dimensionality of the algebra is determined by the number of basis vectors, provides the ring structure and is responsible for the existence of an exact matrix representation. Moreover, the space of functionals is isomorphic to this very linear space, and the algebra of outer products is isomorphic to the algebra of the outer product of these very vectors.

The existence of a unique set of linear independent forms defined at an arbitrary point of the space suggests that the nature of the forms translated over the manifold is similar to the nature of forms which characterize it [7]. This may be also determined by the similar form of the geometric entities as functions of elements of the induced space. Suppose first we have a basis determined by Dirac matrices γ_μ . Making use of this basis, we consider the realization of the wave function in the ordinary Euclidean space. In this case the wave function may be written in terms of a direct sum of a scalar, a vector, a bivector, a trivector, and a pseudoscalar, $\psi = \psi_s \oplus \psi_v \oplus \psi_b \oplus \psi_t \oplus \psi_p$, that is given by

$$\psi = \psi_0 \oplus \psi_\mu \gamma^\mu \oplus \psi_{\mu\nu} \gamma^\mu \gamma^\nu \oplus \psi_{\mu\nu\lambda} \gamma^\mu \gamma^\nu \gamma^\lambda \oplus \psi_{\mu\nu\lambda\rho} \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\rho \quad (1)$$

With the reverse order of composition, we have $\tilde{\psi} = \psi_s \oplus \psi_v \ominus \psi_b \ominus \psi_t \oplus \psi_p$ and having changed the direction of each basis vector, we obtain $\bar{\psi} = \psi_s \ominus \psi_v \oplus \psi_b \ominus \psi_t \oplus \psi_p$. For each even number $\psi = \bar{\psi}$ for $\psi \neq 0$, the wave function

may be reduced to the canonical form [5], i.e.,

$$\psi = \{\rho(x) \exp(i\beta)\}^{\frac{1}{2}} R \quad (2)$$

where $R\tilde{R} = 1$ describes all the coordinate transformations associated with the translation and rotation of coordinates and with the Lorentz transformation in the Euclidean space. This Clifford number can be expressed in terms of an exponential function of the biquaternion $B = q + iq'$, where q and q' are quaternions, each of these representing a sum of a scalar and a dual vector. The physical interpretation of this geometric entity is rather evident since $\rho(x)$ can be associated with the probability density of finding a particle in an arbitrary spatial point, and β is the angle that determines the eigenstate of a particle with positive or negative energy. We have $\beta = 0$ for an electron and $\beta = \pi$ for a positron. Thus it becomes possible to describe the intermediate states of the particle since the form of the wave function of an arbitrary ensemble of particles is analogous.

An arbitrary deformation of the coordinate system can be set in terms of basis deformations $e_\mu = \gamma_\mu R$, where R is the Clifford number that describes arbitrary changes of the basis (including arbitrary displacements and rotations) which do not violate its normalization, i.e., provided $\tilde{R}R = 1$. It is not difficult to verify that $e_\mu^2 = \gamma_\mu \tilde{R} \gamma_\mu R = \gamma_\mu^2 \tilde{R} R = I$ and this does not violate the normalization of the basis [5]. Now, for an arbitrary basis, we can set, at each point of the space, a unique complete linearly independent form as a geometric entity that characterizes this point of the manifold. For a four-dimensional space, such geometric entity may be given by

$$\psi = \psi_0 \oplus \psi_\mu e^\mu \oplus \psi_{\mu\nu} e^\mu e^\nu \oplus \psi_{\mu\nu\lambda} e^\mu e^\nu e^\lambda \oplus \psi_{\mu\nu\lambda\rho} e^\mu e^\nu e^\lambda e^\rho \quad (3)$$

If this point of the manifold is occupied by an elementary particle, then its geometric characteristics may be described by the coefficients of this representation. A product of arbitrary forms of this type is given by a similar form with new coefficients, thus providing the ring structure. The operation of form product may be written as

$$\psi\varphi = \psi \cdot \varphi + \psi \wedge \varphi \quad (4)$$

where $\psi \cdot \varphi$ is an inner product or convolution that decreases the number of basis vectors and $\psi \wedge \varphi$ is outside product that increase number of basis vectors.

In order to determine the operation of form translation over an arbitrary manifold we have to define the derivative operation. It may be written as a linear form $d = e_\mu \frac{\partial}{\partial x_\mu}$ that forms a basis of the vector space of all changes along the curves passing through a given individual point of the space. The action of such an operator on an arbitrary form may be presented as

$$d\psi = d \cdot \psi + d \wedge \psi \quad (5)$$

where $d \cdot \psi$ and $d \wedge \psi$ may be called the "divergence" and the "curl" of the relevant form.

The mapping of the manifold is determined by the mappings of the relevant system of forms. A certain transformation group transforms each form according to the law $\varphi' = \varphi R$, where R determines the mapping elements, of Clifford algebra in our case, and satisfies the condition $\tilde{R}R = 1$. Here action of two successive transformations reduces to the action of the third one, $RP = fQ$, in the Clifford algebra with appropriate structure constants f . In this approach, all the elements of the mapping and all the structure constants are expressed in terms of Clifford numbers of general form with relevant tensor characteristics. For this algebra, we can write the first structure equation that defines the covariant derivative as given by [7]:

$$\Omega = d\psi - \psi\omega \quad (6)$$

with the gauge transformation law for the constraint ω being given by

$$\omega' = R\omega\tilde{R} + R d\tilde{R} \quad (7)$$

Here the tensor representation of the constraint is similar to that of an arbitrary form of the Clifford algebra. In this case the wave function can always be reduced to the canonical form, but local deformations of the proper basis become, however, inobservable since the Tetroude form $R d\tilde{R}$ corresponds to the second term of the gauge transformation. Then the second structure equation that defines the "curvature" form may be written as

$$F = d\omega - \omega\omega \quad (8)$$

with the law of transformation under the algebra being given by $F' = R F \tilde{R}$. This approach makes it possible to consider the mutual relation of fields of different physical nature [7]. However, in what follows we consider only the description of a particle wave function as a geometric entity. The particle wave function is described in terms of a geometric entity with the general representation of the Clifford algebra. The group of possible transformations of the frame of reference must transform the wave function according to the structure equations given above. Under the assumption that the covariant derivative $\Omega \sim m\psi$, the first equation yields that the wave function should be transformed according to the equation

$$d\psi - \psi\omega = m\psi \quad (9)$$

whose form is analogous to the Dirac equation in the spinor representation. The dynamic equation for wave function in geometrical presentation can obtain from action which can present in the term of geometrical invariants as follow:

$$S = \int d\tau \left\{ \Omega \tilde{\Omega} + F \tilde{F} \right\} \quad (10)$$

using the normalization condition $\int d\tau \psi \tilde{\psi} = 1$, where τ is volume of four dimensional space.

This equation is more informative for several reasons. The first one is that spinors are only special projections of Clifford numbers [5], Dirac spinors are represented only by ideals in this algebra, and thus it is impossible to introduce the composition operation on the spinor set. And the most important difference is that complete group of linear transformations of the coordinate system does not exist for spinors [3]. As follows from the previous analysis, a complete transformation group associated with the structure equation exists only in the Clifford-number representation of the wave function. The first structure equation for the wave function reproduces the form of the Dirac equation and, as it has been shown in [5], its solutions are similar to those for the spinor representation. This solves the problem of finite-dimensional representation of the wave function under the complete linear group of coordinate transformations.

For illustration this approach can present the solution of Dirac equation in geometrical presentation. Can consider the behaviour the electron in Coulomb field for relativistic case. In time-space of special relativity theory $d = \gamma_\mu \frac{\partial}{\partial x_\mu}$ a $\omega_\mu \equiv A_\mu$ where $A_\mu = -\gamma_0 \frac{Z\alpha}{r} = \gamma_0 U$ is the vector potential, where α is constant of subtle structure and $m = i\mu\gamma_0$ where μ is mass of electron. The structure equation thus obtained is written in the introduced terms is completely equivalent to the Dirac equation, and has well known solutions both for the calculation of the hydrogen atom spectrum and for the interpretation of electron states [5]. Multiplied the structural equation at the left on γ_0 . After multiplication operator d transform to $\gamma_0 d = \frac{\partial}{\partial x_0} + \nabla$ where spatial gradient $\nabla = \sigma_\mu \frac{\partial}{\partial x_\mu}$ where σ_μ is Pauli matrix. The solution of new equation can present in the form $\psi = (q + q') \exp(iEx_0)$ where q and q' is quaternions, E is energy and $x_0 = ct$ is time coordinate. After this the Dirac equation reduce to two intercoupling equations:

$$\nabla q = (\mu + E - U) q' \quad (11)$$

$$\nabla q' = (\mu - E + U) q \quad (12)$$

The solution of this system equation can be writhe in the form $q = iSG(r)$ and $q' = iPF(r)$ where $G(r)$ and $F(r)$ is number functions and $S = \sigma_r P(\theta, \varphi)$ where $P(\theta, \varphi)$ quaternion which dependence from polar and azimuthal angle in spherical coordinate. Substituted this presentation in system of equation for quaternions can obtain the specter of energy in standard form:

$$E = \mu \left\{ 1 + \frac{(Z\alpha)^2}{\sqrt{l^2 - (Z\alpha)^2} + n_r^2} \right\}^{-\frac{1}{2}} \quad (13)$$

where n_r radial quantum number and l orbital quantum number. Obtained result confirmed the consistency between standard and present approach. The new thing

here is that this equation can be solved for a system with both electric and gravitational fields. We shall not do that since the gravitational field only weakly influences the atomic state and it is very difficult to find the evidence for the geometric nature of the electron. If used the canonical presentation of wave function in the form (2) and assume that $g_{\mu\nu} = \rho(x)G_{\mu\nu}$ where $G_{\mu\nu}$ is potential of repairable gravitational field from action (10) can obtain the system of covariation equations in the form:

$$\theta_{\mu\nu} - \frac{1}{2}\theta g_{\mu\nu} = \frac{1}{6} \left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \right) \quad (14)$$

and

$$\frac{\partial}{\partial x^\mu} \left(\frac{\partial \beta}{\partial x_\mu} \right) \quad (15)$$

where $\theta_{\mu\nu} = \frac{\partial \beta}{\partial x_\mu} \frac{\partial \beta}{\partial x_\nu} - \frac{1}{4}\mu^2 G_{\mu\nu}$, $\theta = \theta_{\mu\nu} g^{\mu\nu}$, $R_{\mu\nu}$ is curvature tensor. This system is whited in presentation $\hbar = c = 1$. For $\beta = \frac{S}{\hbar}$ as standard approach obtained system equations describe the intercoupling between energy-impulse of moving particle with metrical tensor produced it the time-space. Indeed, that since the gravitational field only weakly influences the electron state and it is very difficult to find the evidence for the geometric nature of the electron. Instead we consider a simpler effect, i.e., interference of electrons or other elementary particles whose geometric nature can be revealed by available experimental methods.

We consider the case of particle interference that might be helpful in revealing the geometric character of the wave function. In our representation, two elementary particles can be described by the wave functions represented by geometric entities in the canonical form, i.e., $\psi_1 = \{\rho_1(x) \exp(i\beta_1)\}^{\frac{1}{2}} R_1$ and $\psi_2 = \{\rho_2(x) \exp(i\beta_2)\}^{\frac{1}{2}} R_2$. For electrons we have $\beta_1 = \beta_2 = 0$. The canonical form of the two-electron wave function should be similar, i.e., $\psi = \{\rho(x) \exp(i\beta)\}^{\frac{1}{2}} R = \psi_1 + \psi_2$. Now the post-interference wave function can be written as

$$\psi \tilde{\psi} = \rho(x) = \rho_1 + \rho_2 + (\rho_1 \rho_2)^{\frac{1}{2}} \left\{ R_1 \tilde{R}_2 + \tilde{R}_1 R_2 \right\} \quad (16)$$

In the case of even Clifford numbers, when R corresponds to Lorentz rotations, i.e., when R_i can be written as $R_i = \exp(-B)$, where $B = (\theta + i\varphi)b$ is a double vector, θ and φ are constant numbers, and b is a vector whose modulus is equal to one, the result of interference, for equal particle energies, is given by the standard expression, i.e.,

$$\psi \tilde{\psi} = \rho(x) = \rho_1 + \rho_2 + (\rho_1 \rho_2)^{\frac{1}{2}} \cos \varphi \quad (17)$$

For plane monochromatic waves [5], the solution of the Dirac equation is given by $\psi_1 = \rho_1(x)^{\frac{1}{2}} u \exp(i\sigma_3(p \cdot x))$, where σ_3 is the Pauli matrix, u is particle amplitude, and p is particle momentum. The solution for the second particle is similar except for the phase shift, i.e., we

have $\psi_2 = \rho_2(x)^{\frac{1}{2}} u \exp(i\sigma_3(p \cdot x) + \varphi)$. We see that now electron interference is described by the well known formula. Next we assume that deformations of the reference system are determined by both even and odd numbers and that each transformation contains both even and odd parts, i.e., $R_i = R_i^{ev} + R_i^{od}$, which could be produced by fields of different nature whose effect on different geometric components of the general Clifford number is different [7]. Then the operation

$$R_1 \tilde{R}_2 + \tilde{R}_1 R_2 = R_1^{ev} R_2^{ev} - R_1^{od} R_2^{od} \quad (18)$$

reduces to two terms and we see that the result of electron interference is described by an essentially nonstandard formula. The second term in the right-hand part of the equation also reduces to an even Clifford number, but possesses different structure.

The existence of this effect can be verified experimentally. A coherent electron beam should be divided into two beams, the latter should be passed through separate regions with variable basic geometric characteristics. The change of the wave function passing through different re-

gions can be written as $\psi'_i = \psi_i R_a R_b$ where R_a and R_b describe the transformation of particle characteristics in the regions a and b . If the sequence order is changed, $\psi''_i = \psi_i R_b R_a$ $R_a R_b \neq R_b R_a$ then electron interference should correspond to the last case of the previous analysis, i.e., the interference pattern should be different from the standard case. The various regions can be infinite solenoids of the Aaronov-Bohm experiment with different directions of the magnetic flux. Another way to observe the difference of the interference patterns is to pass electrons along and across the solenoids. The difference is given rise to only by the geometric representation since in the first case the flux is not changed as distinct to the opposite case. We can assume that this effect might be also observed for neutron interference, the regions of variation of the wave function geometric components being two inclusions with different mass numbers occurring on the neutron propagation path. A similar experiment had been proposed in paper [8], however, it has not been performed till now.

-
- [1] W. A. Fock, Zs.for Physics **57**, 2611, (1929).
 - [2] V. A. Zhelnorovich, *Theory of spinors and application in mechanics and physics*, (Nauka, Moskow, 1982).
 - [3] E. Cartan *Lecons sur ia theorie des spineurs*(Actualites scientifiques et industries, Paris, 1938).
 - [4] B. F. Schutz *Geometrical methods of mathematical physics*(Cambridge University Press, Cambridge, 1982).
 - [5] G.Kasanova, *Vector algebr*, (Cambridge University Press, Cambridge,(1979)
 - [6] J. M. Benn and R. W. Tucker, Phys. Lett.A **130**,177 (1983).
 - [7] B. I. Lev, Mod. Phys. Lett. **3**,10, 1025,(1988),ib.**4**, eratum, (1989)
 - [8] A. G. Klein, Physics B, **151**, 44, (1988).
 - [9] J. Lasenby, A. N. Lasenby and J. I. Dosan, Phil Trans. R. Soc.Lond., 1, (1996)